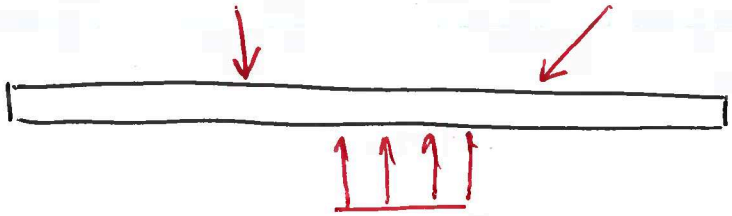
Lecture 10
22-Oct-2026



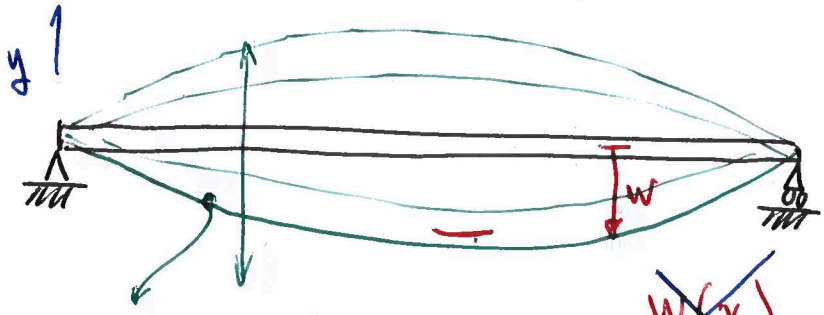
BEAM



BAR



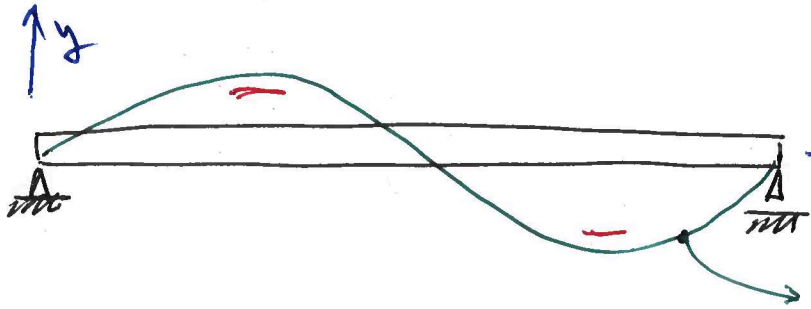
SHAFT



1st mode shape

~~w(x)~~ → w(x,t)

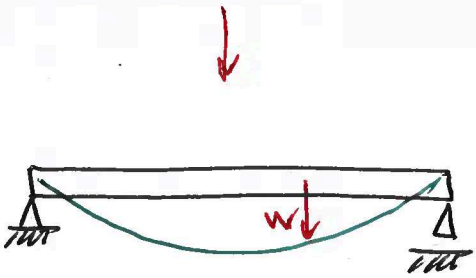
statics → dynamics



2nd mode shape



3rd mode shape



Rayleigh's method

A.M.M.



$$\omega = \sqrt{\frac{\bar{k}}{\bar{M}}}$$

Now do we find $\bar{M}$ and $\bar{k}$ for the continuous system



Lagrange EoM

$\bar{T}$ and V_e and V_g

$$\bar{T} = \frac{1}{2} m v^2 = \frac{1}{2} \int_0^L \rho A \left(\frac{\partial w}{\partial t}\right)^2 dx$$

$\dot{w}$ = velocity

mass ($\rho A L$)

$$= \frac{1}{2} \rho A \int_0^L \left(\frac{\partial w}{\partial t}\right)^2 dx$$

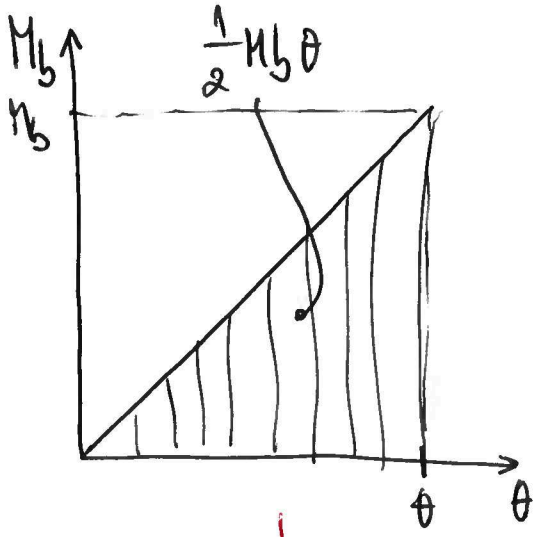
Potential energy $V = V_e + V_g$

$$V_g = mgh = g\int_0^L \rho A w\, dx = g\rho A \int_0^L w\, dx$$

"h"

mass (ρAL)

$$V_e = \frac{1}{2} k \Delta x^2 = \frac{1}{2}\int_0^L M_b\, d\theta$$

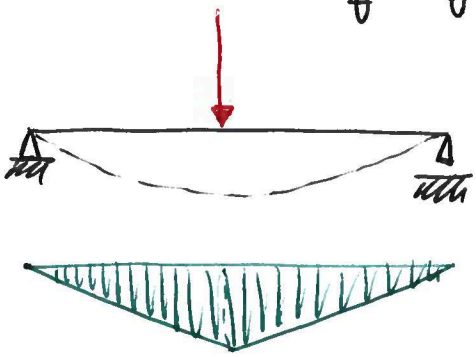


$$M_b = EI \frac{\partial^2 w}{\partial x^2}$$

$$V_e = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right) d\theta$$

$$V_e = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) dx$$

$$V_e = \frac{1}{2} EI \int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$



$$\theta = \frac{\partial w}{\partial x}$$

$$\frac{d\theta}{dx} = \frac{\partial^2 w}{\partial x^2}$$

$$d\theta = \frac{\partial^2 w}{\partial x^2} dx$$

$w(x,t)$

Time derivatives (t)

$\dot{w}$

$\ddot{w}$

Space derivatives (x)

w'

w''

w'''

w''''

KE

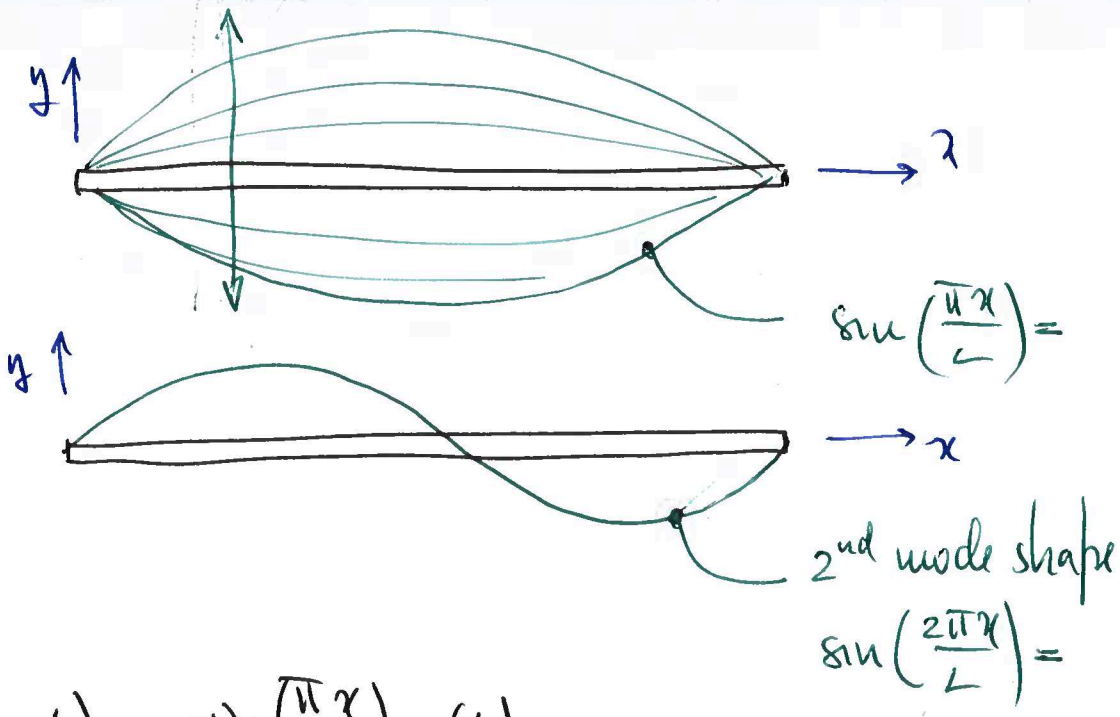
$$T = \frac{1}{2}\rho A \int_0^L \dot{w}^2 \, dx$$

PE

$$V_e = \frac{1}{2} EI \int_0^L (w'')^2 dx$$

$$V_g = \rho A g \int_0^L w \, dx$$

$w(x,t)$??



$$w_1(x,t) = \sin\left(\frac{\pi x}{L}\right) q(t)$$

$w(x,t) = ?$

- Travelling wave: $w(x,t) = A\sin(x-ct)$
- Standing waves: $w(x,t) = \varphi(x)\, q(t)$

$$w(x,t) = \varphi(x)\, q(t)$$

- $\varphi(x)$: shape function (mode shape)
- $q(t)$: generalised coordinate (Amplitude function)



Kinetic energy

$$T = \frac{1}{2}\rho A \int_0^L \dot{w}^2 dx = \frac{1}{2}\rho A \int_0^L \varphi^2 \dot{q}^2 dx$$

$w = \varphi q$

$\dot{w} = \varphi \dot{q}$

$$T = \frac{1}{2}\rho A \dot{q}^2 \int_0^L \varphi^2 dx = T(t)$$

$\int_0^L \varphi^2 dx$ → constant

Potential energy

$$V_e = \frac{1}{2} EI \int_0^L (w'')^2 dx = \frac{1}{2} EI \int_0^L \varphi''^2 q^2 dx$$

$w = \varphi q$

$w' = \varphi' q$

$w'' = \varphi'' q$

$$V_e = \frac{1}{2} EI\, q^2 \int_0^L (\varphi'')^2 dx = V_e(t)$$

$\int_0^L (\varphi'')^2 dx$ → constant

$$V_g = \rho A g \int_0^L w\, dx = \rho A g \int_0^L \varphi q\, dx$$

$w = \varphi q$

$$V_g = \rho A g\, q \int_0^L \varphi\, dx = V_g(t)$$

$\int_0^L \varphi\, dx$ → constant

$T(t)$, $V_e(t)$, $V_g(t)$

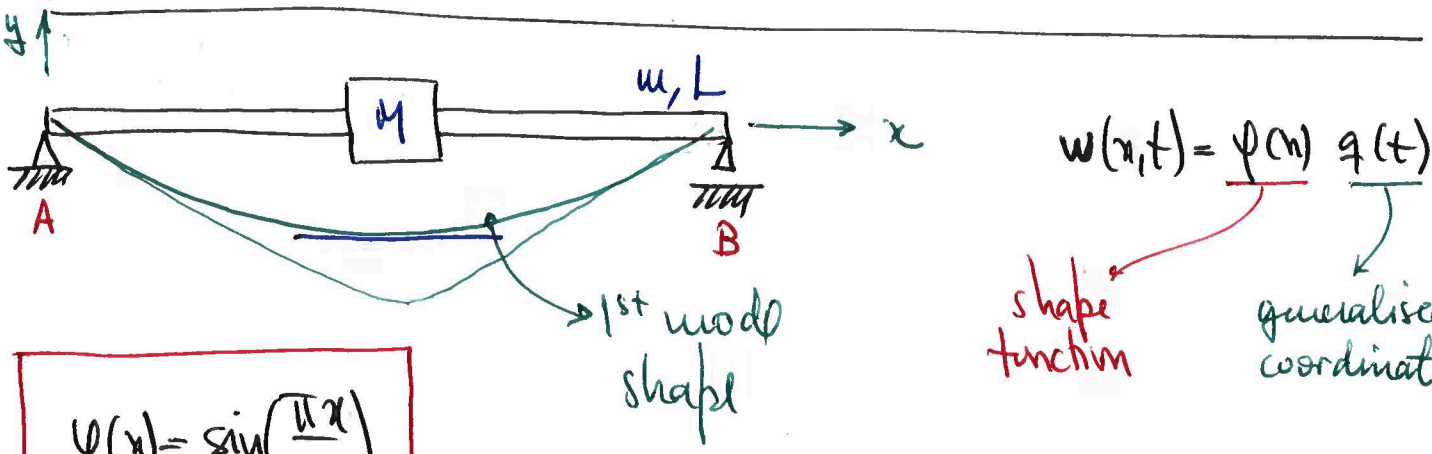
→ Lagrange

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q_q}$$

→ EoM $\quad \boxed{\ldots \ddot{q} + \ldots q + \ldots \dot{q} + \ldots = \ldots}$

effective inertia $\bar{M}\ddot{q}$ + effective stiffness $\bar{k}q$ + $C\dot{q} + \ldots = \ldots$

$$\omega = \sqrt{\frac{\bar{k}}{\bar{M}}}$$



$$w(x,t) = \varphi(x)\, q(t)$$

shape function — generalised coordinate

$$\boxed{\varphi(x) = \sin\left(\frac{\pi x}{L}\right)}$$

BC_A $(x=0) \Rightarrow w(0,t) = 0 \Rightarrow \varphi(x) = 0$ ✓

$(x=0) \Rightarrow w''(0,t) = 0 \Rightarrow \varphi''(0) = 0$ ✓

BC_B $(x=L) \Rightarrow w(L,t) = 0 \Rightarrow \varphi(L) = 0$ ✓

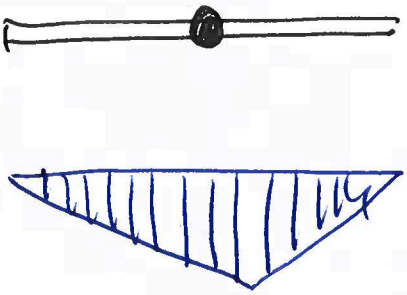
$(x=L) \Rightarrow w''(L,t) = 0 \rightarrow \varphi''(L) = 0$ ✓

$(BC_{1/2L}$ $(x=1/2) \Rightarrow w'\left(\frac{L}{2},t\right) = 0 \Rightarrow \varphi'\left(\frac{L}{2}\right) = 0$ ✓ $)$

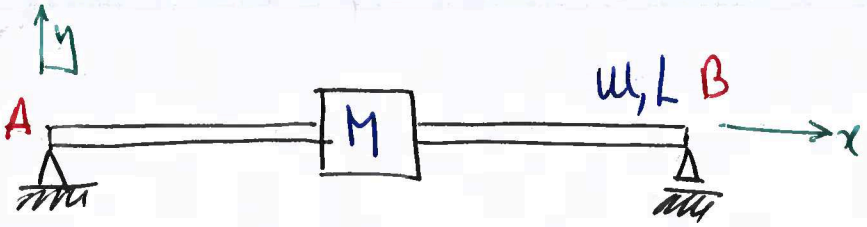
EoM ⇝ 

⇝ $\omega = \sqrt{\frac{\bar{k}}{\bar{M}}}$

$$M_b = EI\, w''$$



Lecture 13
3-Nov-2026



$$w(x,t) = \varphi(x)\, q(t)$$

$$\varphi(x) = \sin\frac{\pi x}{L}$$

$$T = \frac{1}{2}\rho A \dot{q}^2 \int_0^L \varphi^2 dx + \frac{1}{2} M \dot{w}\left(\frac{L}{2},t\right)^2$$

KE beam

KE lumped mass

$$w(x,t) = \sin\left(\frac{\pi x}{L}\right) q(t)$$

$$\dot{w}(x,t) = \sin\left(\frac{\pi x}{L}\right)\dot{q}$$

$$\dot{w}\left(\frac{L}{2},t\right) = \sin\left(\frac{\pi L}{2L}\right)\dot{q} = \dot{q}$$

$$\int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\Big]_0^L$$

$$= \frac{L}{2} - 0 - (0-0) = \frac{L}{2}$$

$$T = \frac{1}{2}\rho A \dot{q}^2 \frac{L}{2} + \frac{1}{2} M \dot{q}^2 = \boxed{\frac{1}{2}\left(M + \frac{m}{2}\right)\dot{q}^2 = T}$$

m

$$V = \text{elastic (bending) beam} + \text{grav. } M + \text{grav. } m \text{ (beam)}$$

$$= \frac{1}{2} EI \int_0^L (w'')^2 dx + Mg\, w\left(\frac{L}{2},t\right) + \rho A g \int_0^L w\, dx$$

① ② ③

- $w' = \varphi' q = \frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) q$

$$w'' = \varphi'' q = -\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) q$$

$$(w'')^2 = \left(\frac{\pi}{L}\right)^4 \sin^2\left(\frac{\pi x}{L}\right) q^2$$

- $w\left(\frac{L}{2},t\right) = \underbrace{\sin\left(\frac{\pi L}{2L}\right)}_{1} q = q$

- $\int_0^L w\,dx = -\frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)q\Big]_0^L = -\frac{L}{\pi}(-1-1)q = \frac{2L}{\pi}q$

$$V = \frac{1}{2}EI\left(\frac{\pi}{L}\right)^4\frac{L}{2}q^2 + Mgq + \rho A g q\left(\frac{2L}{\pi}\right)$$

$\rho A = m$

↳ Lagrange : $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q_q$

$\frac{\partial T}{\partial q} = 0$, $Q_q = 0$

c…)

$$\left(M+\frac{m}{2}\right)\ddot q + EI\left(\frac{\pi}{L}\right)^4\frac{L}{2}q + \left(Mg + \frac{2}{\pi}mg\right) = 0$$

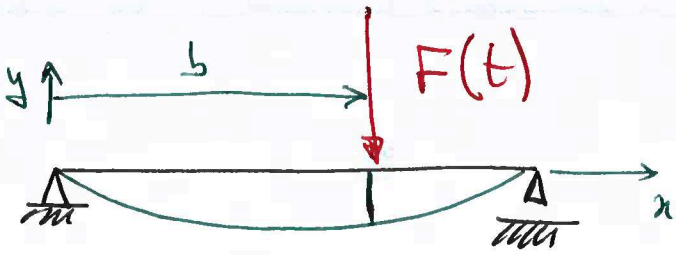
$$\left(M+\frac{m}{2}\right)\ddot q + EI\frac{\pi^4}{2L^3}q + \left(M+\frac{2m}{\pi}\right)g = 0$$

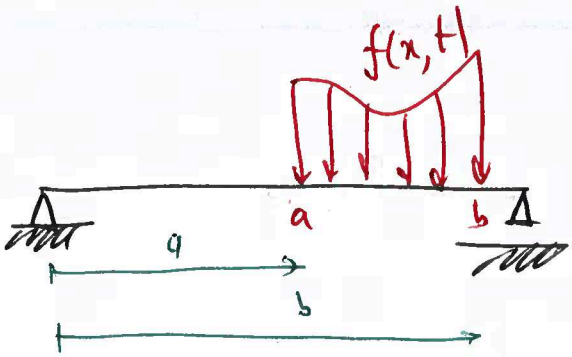
$\overline{M}\,\ddot q + \overline{k}\,q + \dots = \dots$



$$\omega = \sqrt{\frac{\overline{k}}{\overline{M}}}$$

$$\omega = \sqrt{\frac{\overline{k}}{\overline{M}}} = \sqrt{\left(EI\frac{\pi^4}{2L^3}\right)\Big/\left(M+\frac{m}{2}\right)} = \frac{\pi^2}{\pi}\sqrt{\frac{EI}{(2M+m)L^3}}$$

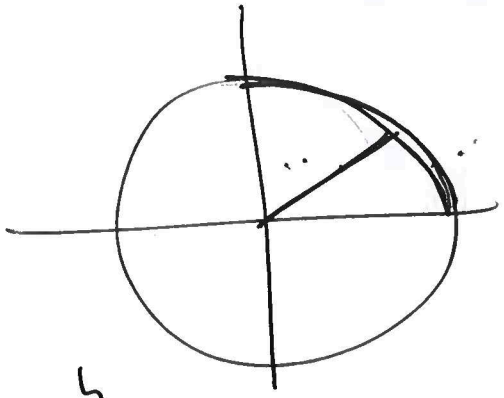




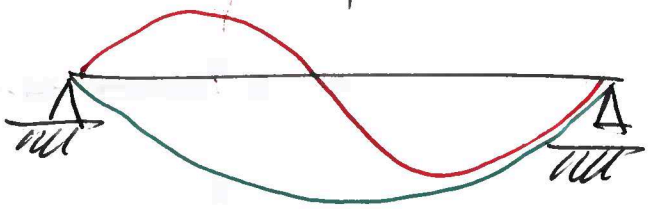
$$Q = \varphi(b)\, F(t)$$

$$[-]$$

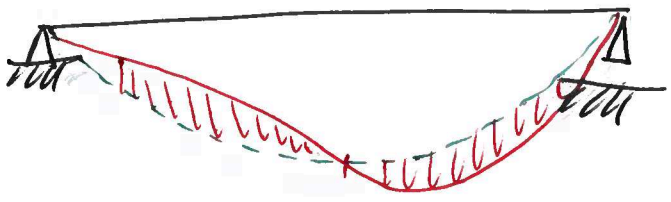
x^2 $\quad$ $\sin\frac{\pi x}{L}$ $\quad$ rad



$$Q = \int_a^b \varphi(x)\, f(x,t)\, dx$$







$$w(x,t) = \sum_i^{\infty} \varphi_i(x)\, q_i(t)$$